Solution to Problem Set 8

1. (a) Using information from the table, we have

$$\int_0^2 [3f'(x) - 4xg'(x^2)]dx = 3\int_0^2 f'(x)dx - 2\int_0^2 g'(x^2)dx^2$$
$$= 3[f(x)]_0^2 - 2[g(x^2)]_0^2 = 3[f(2) - f(0)] - 2[g(4) - g(0)]$$
$$= 3(0-2) - 2(1-2) = -4.$$

(b) Using information from the table, we have

$$\int_{-1}^1 [f(x+3)]^3 f'(x+3)dx = \int_{-1}^1 [f(x+3)]^3 df(x+3)$$
$$= \left[\frac{1}{4}[f(x+3)]^4\right]_{-1}^1 = \frac{1}{4}[f(4)]^4 - \frac{1}{4}[f(2)]^4$$
$$= \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4 = \frac{1}{4}.$$

2. With the substitution $u = \frac{1}{x}$, we have $x = \frac{1}{u}$, $dx = -\frac{1}{u^2}du$, $u = 2$ when $x = \frac{1}{2}$, and $u = \frac{1}{2}$ when $x = 2$. Thus,

$$\int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2}dx = \int_2^{\frac{1}{2}} \frac{\ln\left(\frac{1}{u}\right)}{1+\left(\frac{1}{u}\right)^2}\left(-\frac{1}{u^2}\right)du = -\int_2^{\frac{1}{2}} \frac{-\ln u}{u^2+1}du = -\int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2}dx.$$

Rearranging the above equation we get $2\int_{1/2}^2 \frac{\ln x}{1+x^2}dx = 0$, so

$$\int_{\frac{1}{2}}^2 \frac{\ln x}{1+x^2}dx = 0.$$

3. (a) Let $u = a - x$. Then $du = -dx$, $u = a$ when $x = 0$, and $u = 0$ when $x = a$. Thus we have

$$\int_0^a f(a-x)dx = -\int_a^0 f(u)du = \int_0^a f(u)du = \int_0^a f(x)dx.$$

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(b) (i) Using the result from (a) with $a = \frac{\pi}{2}$ and $f(x) = (\cos x - \sin x)^{1013}$, we have

$$\int_0^{\frac{\pi}{2}} (\cos x - \sin x)^{1013}dx = \int_0^{\frac{\pi}{2}} \left[\cos\left(\frac{\pi}{2} - x\right) - \sin\left(\frac{\pi}{2} - x\right)\right]^{1013} dx$$
$$= \int_0^{\frac{\pi}{2}} (\sin x - \cos x)^{1013}dx = -\int_0^{\frac{\pi}{2}} (\cos x - \sin x)^{1013}dx.$$

This gives $2\int_0^{\pi/2} (\cos x - \sin x)^{1013}dx = 0$, and so

$$\int_0^{\frac{\pi}{2}} (\cos x - \sin x)^{1013}dx = 0.$$

(ii) Using the result from (a) with $a = \pi$ and $f(x) = x\sin^2 x$, we have

$$\int_0^{\pi} x\sin^2 x\,dx = \int_0^{\pi} (\pi - x)\sin^2(\pi - x)\,dx = \pi\int_0^{\pi}\sin^2 x\,dx - \int_0^{\pi} x\sin^2 x\,dx\,,$$

This gives $2\int_0^{\pi} x\sin^2 x\,dx = \pi\int_0^{\pi}\sin^2 x\,dx$, and so

$$\int_0^{\pi} x\sin^2 x\,dx = \frac{1}{2}\left(\pi\int_0^{\pi}\sin^2 x\,dx\right) = \frac{\pi}{2}\int_0^{\pi}\frac{1-\cos 2x}{2}dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{4}.$$

(iii) Using the result from (a) with $a = \frac{\pi}{2}$ and $f(x) = \frac{\cos^3 x}{\sin x + \cos x}$, we have

$$\int_0^{\frac{\pi}{2}}\frac{\cos^3 x}{\sin x + \cos x}dx = \int_0^{\frac{\pi}{2}}\frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}dx = \int_0^{\frac{\pi}{2}}\frac{\sin^3 x}{\cos x + \sin x}dx.$$

Now

$$\int_0^{\frac{\pi}{2}}\frac{\cos^3 x}{\sin x + \cos x}dx + \int_0^{\frac{\pi}{2}}\frac{\sin^3 x}{\sin x + \cos x}dx$$

$$= \int_0^{\frac{\pi}{2}}\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x}dx = \int_0^{\frac{\pi}{2}}(\sin^2 x - \sin x\cos x + \cos^2 x)dx$$

$$= \int_0^{\frac{\pi}{2}}\left(1 - \frac{1}{2}\sin 2x\right)dx = \left[x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - \frac{1}{4}\right) - \left(0 + \frac{1}{4}\right) = \frac{\pi - 1}{2},$$

so

$$\int_0^{\frac{\pi}{2}}\frac{\cos^3 x}{\sin x + \cos x}dx = \frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\frac{\cos^3 x}{\sin x + \cos x}dx + \int_0^{\frac{\pi}{2}}\frac{\sin^3 x}{\sin x + \cos x}dx\right) = \frac{\pi - 1}{4}.$$

4. We have

$$\int_0^{\frac{1}{2}}\frac{t^2 + 2t + 3}{\sqrt{1 - t^2}}dt = \int_0^{\frac{1}{2}}\frac{(t^2 - 1) + 2t + 4}{\sqrt{1 - t^2}}dt = \int_0^{\frac{1}{2}}\frac{t^2 - 1}{\sqrt{1 - t^2}}dt + \int_0^{\frac{1}{2}}\frac{2t}{\sqrt{1 - t^2}}dt + \int_0^{\frac{1}{2}}\frac{4}{\sqrt{1 - t^2}}dt$$

$$= -\int_0^{\frac{1}{2}}\sqrt{1 - t^2}dt + \int_0^{\frac{1}{2}}\frac{2t}{\sqrt{1 - t^2}}dt + 4\int_0^{\frac{1}{2}}\frac{1}{\sqrt{1 - t^2}}dt.$$

Now

⊙ Similar to Q6(c) of Problem Set 7, $\int_0^{1/2}\sqrt{1 - t^2}dt$ represents the total area of a circular sector of angle $\frac{\pi}{6}$ and of radius 1 centered at the origin, together with a right-angled triangle beneath it. So

$$\int_0^{\frac{1}{2}}\sqrt{1 - t^2}dt = \underbrace{\frac{\pi/6}{2\pi}\cdot\pi(1)^2}_{\text{area of sector}} + \underbrace{\frac{1}{2}\left(\frac{1}{2}\right)\sqrt{1 - \left(\frac{1}{2}\right)^2}}_{\text{area of triangle}} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}.$$

- With a substitution $u = 1 - t^2$, we have $du = -2t\,dt$, $u = 1$ when $t = 0$ and $u = \frac{3}{4}$ when $t = \frac{1}{2}$. So

$$\int_0^{\frac{1}{2}} \frac{2t}{\sqrt{1-t^2}}\,dt = -\int_1^{\frac{3}{4}} \frac{1}{\sqrt{u}}\,du = \int_{\frac{3}{4}}^{1} \frac{1}{\sqrt{u}}\,du = \left[2\sqrt{u}\right]_{\frac{3}{4}}^{1} = 2 - \sqrt{3}.$$

- We also have

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-t^2}}\,dt = [\arcsin t]_0^{\frac{1}{2}} = \frac{\pi}{6}.$$

Therefore

$$\int_0^{\frac{1}{2}} \frac{t^2+2t+3}{\sqrt{1-t^2}}\,dt = -\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right) + \left(2 - \sqrt{3}\right) + 4 \cdot \frac{\pi}{6} = 2 - \frac{9}{8}\sqrt{3} + \frac{7\pi}{12}.$$

5. (a) (i) Let $u = a - x$. Then $du = -dx$, $u = a$ when $x = 0$, and $u = 0$ when $x = a$. Thus we have

$$\int_0^a f(a-x)dx = -\int_a^0 f(u)du = \int_0^a f(u)du = \int_0^a f(x)dx.$$

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(ii) Suppose that $f(x) + f(a-x) = c$ for every $x \in [0,a]$. Then with $x = \frac{a}{2}$ we have $f\left(\frac{a}{2}\right) + f\left(\frac{a}{2}\right) = c$, so

$$f\left(\frac{a}{2}\right) = \frac{c}{2}.$$

Moreover, we also have

$$\int_0^a f(x)dx + \int_0^a f(a-x)dx = \int_0^a \big(f(x) + f(a-x)\big)dx = \int_0^a c\,dx = ac.$$

But the two integrals on the left are the same according to (a) (i), so

$$\int_0^a f(x)dx = \frac{ac}{2} = af\left(\frac{a}{2}\right).$$

(b) Let $f\colon [0, 2\pi] \to \mathbb{R}$ be the continuous function

$$f(x) = \frac{1}{e^{\sin^3 x} + 1}.$$

Then we can verify that

$$f(x) + f(2\pi - x) = \frac{1}{e^{\sin^3 x} + 1} + \frac{1}{e^{\sin^3(2\pi - x)} + 1} = \frac{1}{e^{\sin^3 x} + 1} + \frac{1}{e^{-\sin^3 x} + 1}$$
$$= \frac{1}{e^{\sin^3 x} + 1} + \frac{e^{\sin^3 x}}{1 + e^{\sin^3 x}} = \frac{1 + e^{\sin^3 x}}{1 + e^{\sin^3 x}} = 1$$

for every $x \in [0, 2\pi]$. Thus applying the result from (a) (ii), we have

$$\int_0^{2\pi} \frac{1}{e^{\sin^3 x} + 1}\,dx = 2\pi f(\pi) = 2\pi \cdot \frac{1}{e^{\sin^3 \pi} + 1} = \pi.$$

6. (a) Using the double angle formula twice, we have

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)dx$$

$$= \int \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx = \frac{3}{8}\int dx - \frac{1}{2}\int \cos 2x\,dx + \frac{1}{8}\int \cos 4x\,dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C,$$

where C is an arbitrary constant.

(b) The product-to-sum formula gives $\cos 1013x \cos 2025x = \frac{1}{2}(\cos 3038x + \cos 1012x)$, so

$$\int_{-\pi}^{\pi} \cos 1013x \cos 2025x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos 3038x\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos 1012x\,dx$$

$$= \frac{1}{2}\left[\frac{1}{3038}\sin 3038x\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{1}{1012}\sin 1012x\right]_{-\pi}^{\pi} = 0.$$

(c) Whenever x is not an integer multiple of $\frac{\pi}{2}$, we have

$$\frac{\sin 3x}{\sin x} + \frac{\cos 3x}{\cos x} = \frac{\sin 3x \cos x + \cos 3x \sin x}{\sin x \cos x} = \frac{\sin(3x+x)}{\frac{1}{2}\sin 2x}$$

$$= \frac{\sin 4x}{\frac{1}{2}\sin 2x} = \frac{2\sin 2x \cos 2x}{\frac{1}{2}\sin 2x} = 4\cos 2x\,,$$

so

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\sin 3x}{\sin x} + \frac{\cos 3x}{\cos x}\right)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\cos 2x\,dx = [2\sin 2x]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 2 - \sqrt{3}.$$

(d) Let $u = \cos x$. Then $du = -\sin x\,dx$, $u = 1$ when $x = 0$, and $u = 0$ when $x = \frac{\pi}{2}$. So

$$\int_0^{\frac{\pi}{2}} \sin^5 x\,dx = \int_0^{\frac{\pi}{2}} -\sin^4 x\,(-\sin x\,dx) = \int_0^{\frac{\pi}{2}} -(1-\cos^2 x)^2(-\sin x\,dx)$$

$$= \int_1^0 -(1-u^2)^2 du = \int_0^1 (1 - 2u^2 + u^4)du$$

$$= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \frac{8}{15}.$$

(e) Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\int \sec^4 x \tan^2 x\,dx = \int \sec^2 x \tan^2 x\,(\sec^2 x\,dx) = \int (1 + \tan^2 x)\tan^2 x\,d\tan x$$

$$= \int (\tan^2 x + \tan^4 x)d\tan x = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C,$$

where C is an arbitrary constant.

7. (a) The mean voltage (in V) over 1 second is given by

$$\frac{1}{1-0}\int_0^1 E(t)dt = \int_0^1 311\sin(100\pi t)\,dt = \left[-\frac{311}{100\pi}\cos(100\pi t)\right]_0^1 = 0.$$

(b) First we have

$$\int_0^1 \left(E(t)\right)^2 dt = \int_0^1 311^2\sin^2(100\pi t)\,dt = 311^2\int_0^1 \frac{1-\cos(200\pi t)}{2}dt$$

$$= 311^2\left[\frac{1}{2}t - \frac{1}{400\pi}\sin(200\pi t)\right]_0^1 = \frac{311^2}{2}.$$

The RMS voltage (in V) over 1 second is given by

$$\sqrt{\frac{1}{1-0}\int_0^1 \left(E(t)\right)^2 dt} = \sqrt{\frac{311^2}{2}} = \frac{311}{\sqrt{2}} \approx 220.$$

(c) Suppose that the amplitude of the voltage function is A (in V). Then the voltage function becomes

$$E(t) = A\sin(100\pi t).$$

In order that the RMS voltage is 110 V, we require

$$110 = \sqrt{\frac{1}{1-0}\int_0^1 \left(E(t)\right)^2 dt} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}}.$$

Therefore the amplitude of the voltage function (in V) is $110\sqrt{2} \approx 156$.

8. (a) Assuming that the equator has radius 1, the circle on the Earth surface with latitude ϕ has radius $1\cdot\cos\phi$. So its circumference is

$$2\pi(1\cdot\cos\phi) = 2\pi\cos\phi.$$

Now this circumference is unrolled to become a horizontal line segment of length 2π, so it is stretched by a factor of

$$\text{stretching factor} = \frac{\text{new length}}{\text{original length}} = \frac{2\pi}{2\pi\cos\phi} = \sec\phi.$$

(b) The function $y(\phi)$ satisfies the differential equation $y'(\phi) = \sec\phi$, so

$$y(\phi) = \int \sec\phi\,d\phi = \int \frac{\sec\phi\,(\sec\phi + \tan\phi)}{\sec\phi + \tan\phi}d\phi$$

$$= \int \frac{\sec\phi\tan\phi + \sec^2\phi}{\sec\phi + \tan\phi}d\phi = \ln|\sec\phi + \tan\phi| + C,$$

where C is some constant. Since the equator has zero latitude and zero displacement from itself, we must have $y(0) = 0$. So we have $0 = \ln|\sec 0 + \tan 0| + C$, and so $C = 0$. Therefore

$$y(\phi) = \ln|\sec\phi + \tan\phi|.$$

(c) If the length and width of the map are both 2π, then the distance from the top of the map to the equator is π. Since the map covers all points with latitude in $[-\alpha, \alpha]$, we must have $y(\alpha) = \pi$, i.e.

$$\ln|\sec\alpha + \tan\alpha| = \pi.$$

On solving this equation for $\alpha \in (0, 90°)$ we have $\sec\alpha + \tan\alpha = e^{\pi}$, i.e.

$$1 + \sin\alpha = e^{\pi}\cos\alpha$$

$$\frac{e^{\pi}}{\sqrt{1+e^{2\pi}}}\cos\alpha - \frac{1}{\sqrt{1+e^{2\pi}}}\sin\alpha = \frac{1}{\sqrt{1+e^{2\pi}}}$$

$$\cos\left(\alpha + \arcsin\frac{1}{\sqrt{1+e^{2\pi}}}\right) = \frac{1}{\sqrt{1+e^{2\pi}}}$$

which gives the solution

$$\alpha = \arccos\frac{1}{\sqrt{1+e^{2\pi}}} - \arcsin\frac{1}{\sqrt{1+e^{2\pi}}} \approx 85.05°.$$

9. (a) Let $u = \arctan\sqrt{x}$ and $v = 2\sqrt{x}$ (so that $dv = \frac{1}{\sqrt{x}}dx$). Then $du = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}dx = \frac{1}{2\sqrt{x}(1+x)}dx$, so antidifferentiation by parts gives

$$\begin{aligned}\int \frac{1}{\sqrt{x}}\arctan\sqrt{x}\,dx &= (\arctan\sqrt{x})(2\sqrt{x}) - \int (2\sqrt{x})\frac{1}{2\sqrt{x}(1+x)}dx \\ &= 2\sqrt{x}\arctan\sqrt{x} - \int\frac{1}{1+x}dx \\ &= 2\sqrt{x}\arctan\sqrt{x} - \ln|1+x| + C \\ &= 2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C,\end{aligned}$$

where C is an arbitrary constant.

(b) We first expand the integrand to get

$$\int_0^{\pi}(x+\sin x)^2dx = \int_0^{\pi}(x^2 + 2x\sin x + \sin^2 x)dx = \int_0^{\pi}x^2dx + \int_0^{\pi}2x\sin x\,dx + \int_0^{\pi}\sin^2 x\,dx.$$

Now we have

- $\int_0^{\pi}x^2dx = \left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{1}{3}\pi^3$,
- $\int_0^{\pi}\sin^2 x\,dx = \int_0^{\pi}\frac{1-\cos 2x}{2}dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}$, and
- $\int_0^{\pi}2x\sin x\,dx = [-2x\cos x]_0^{\pi} - \int_0^{\pi}-2\cos x\,dx = [-2x\cos x]_0^{\pi} + [2\sin x]_0^{\pi} = 2\pi.$

Therefore

$$\int_0^{\pi}(x+\sin x)^2dx = \frac{1}{3}\pi^3 + 2\pi + \frac{\pi}{2} = \frac{1}{3}\pi^3 + \frac{5}{2}\pi.$$

(c) Let $u = 1 + x$. Then $x = u - 1$ and $du = dx$, so

$$\int x^2 \ln(1+x)\,dx = \int (u-1)^2 \ln u\,du = (\ln u)\left(\frac{1}{3}(u-1)^3\right) - \int \frac{1}{3}(u-1)^3\left(\frac{1}{u}du\right)$$

$$= \frac{1}{3}(u-1)^3 \ln u - \frac{1}{3}\int \frac{u^3 - 3u^2 + 3u - 1}{u}\,du$$

$$= \frac{1}{3}(u-1)^3 \ln u - \frac{1}{3}\int \left(u^2 - 3u + 3 - \frac{1}{u}\right)du$$

$$= \frac{1}{3}(u-1)^3 \ln u - \frac{1}{9}u^3 + \frac{1}{2}u^2 - u + \frac{1}{3}\ln u + C$$

$$= \frac{1}{3}x^3 \ln(1+x) - \frac{1}{9}(1+x)^3 + \frac{1}{2}(1+x)^2 - (1+x) + \frac{1}{3}\ln(1+x) + C$$

$$= \frac{1}{3}x^3 \ln(1+x) - \frac{1}{9}x^3 + \frac{1}{6}x^2 - \frac{1}{3}x + \frac{1}{3}\ln(1+x) + C,$$

where C is an arbitrary constant.

(d) Integration by parts gives

$$\int_1^e \sin(\ln x)\,dx = [x\sin(\ln x)]_1^e - \int_1^e x\cdot\cos(\ln x)\cdot\frac{1}{x}dx = [x\sin(\ln x)]_1^e - \int_1^e \cos(\ln x)\,dx$$

$$= [x\sin(\ln x)]_1^e - [x\cos(\ln x)]_1^e + \int_1^e x\cdot -\sin(\ln x)\cdot\frac{1}{x}dx$$

$$= e\sin 1 - e\cos 1 + 1 - \int_1^e \sin(\ln x)\,dx\,.$$

Rearranging the terms, we get

$$\int_1^e \sin(\ln x)\,dx = \frac{1}{2}(e\sin 1 - e\cos 1 + 1).$$

10. The daily sales can be regarded as the "rate of change" of the total sales. Thus the total sales for the first 100 days can be estimated as the integral of the daily sales from 0 to 100, i.e.

$$\int_0^{100} S(t)dt = \int_0^{100} 100t^2e^{-\frac{t}{5}}dt = \left[(100t^2)\left(-5e^{-\frac{t}{5}}\right)\right]_0^{100} - \int_0^{100}\left(-5e^{-\frac{t}{5}}\right)(200t\,dt)$$

$$= \left[-500t^2e^{-\frac{t}{5}}\right]_0^{100} + \int_0^{100} 1000te^{-\frac{t}{5}}dt$$

$$= \left[-500t^2e^{-\frac{t}{5}}\right]_0^{100} + \left[(1000t)\left(-5e^{-\frac{t}{5}}\right)\right]_0^{100} - \int_0^{100}\left(-5e^{-\frac{t}{5}}\right)(1000\,dt)$$

$$= \left[-500t^2e^{-\frac{t}{5}}\right]_0^{100} + \left[-5000te^{-\frac{t}{5}}\right]_0^{100} + \int_0^{100} 5000e^{-\frac{t}{5}}dt$$

$$= \left[-500t^2e^{-\frac{t}{5}}\right]_0^{100} + \left[-5000te^{-\frac{t}{5}}\right]_0^{100} + \left[-25000e^{-\frac{t}{5}}\right]_0^{100}$$

$$= -500(100)^2e^{-20} - 5000(100)e^{-20} - 25000e^{-20} + 25000$$

$$\approx 25000.$$

11. (a) The final displacement of the vehicle is the integral of the velocity from $t = 0$ to $t = 40$. This is equal to the net area under the given graph on the interval $[0, 40]$, which is

$$\int_0^{40} v(t)dt = \frac{1}{2}(10)(10) + \left(10^2 - \frac{1}{4}\pi(10)^2\right) - \frac{1}{2}(10)(10) - \frac{1}{4}\pi(10)^2$$

$$= 100\left(1 - \frac{\pi}{2}\right) \approx -57.08.$$

So the final displacement of the vehicle from its initial position is about -57.08 m.

(b) The vehicle first goes forward from $t = 0$ to $t = 20$, and then backward from $t = 20$ to $t = 40$. The furthest displacement happens when $t = 20$, which is

$$\int_0^{20} v(t)dt = \frac{1}{2}(10)(10) + \left(10^2 - \frac{1}{4}\pi(10)^2\right)$$

$$= 100\left(\frac{3}{2} - \frac{\pi}{4}\right) \approx 71.46.$$

So the furthest displacement of the vehicle from its initial position is about 71.46 m.

(c) The total distance traveled by the vehicle is the integral $\int_0^T |v(t)|dt$. This is equal to the actual area bounded between the graph and the horizontal axis, on the interval $[0, T]$, which is

$$\int_0^T |v(t)|dt = \frac{1}{2}(10)(10) + \left(10^2 - \frac{1}{4}\pi(10)^2\right) + \frac{1}{2}(10)(10) + \frac{1}{4}\pi(10)^2$$

$$= 200.$$

So the total distance traveled by the vehicle is 200 m.